University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 2
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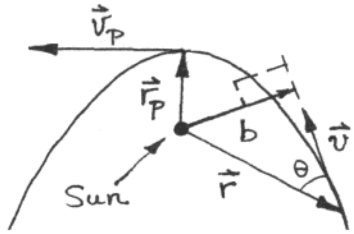
Reading:
105 Notes 2.1, 2.2, 2.3, 2.4, 2.5.
Hand & Finch pp. 10-12, 130-134, 284-285.

1.
A comet, barely unbound by the sun (its total energy vanishes), executes a parabolic orbit about it.



At a certain time the comet has speed v and impact parameter b with respect to the sun. You may neglect the comet's mass m with respect to the sun's mass M. Find the perigee (distance of closest approach to the sun) of the comet.
Solution:
At the initial time, the angular momentum is

$$l = |\mathbf{r} \times \mathbf{p}| = mvr|\sin\theta| = mvb\ ,$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{v}$, and $b \equiv r|\sin\theta|$. At the perigee, $l = mv_p r_p$, where v_p and r_p are the velocity and radius there. So, by angular momentum conservation, $v_p = vb/r_p$.

What about energy? As the problem states, a "barely bound" orbit is one with zero total energy, *i.e.* with kinetic energy exactly balancing potential energy. So we can set the energy at the perigee equal to zero:

$$\tfrac{1}{2}mv_p^2 - \frac{GMm}{r_p} = 0\ .$$

Substitute for v_p:

$$\frac{v^2b^2}{2r_p^2} = \frac{GM}{r_p}\ .$$

Solving for r_p,

$$r_p = \frac{v^2b^2}{2GM}\ .$$

2.
Two masses m_1 and m_2 orbit around their common center of mass (CM), which has the coordinate $\mathbf{R}(t)$. They are separated from the CM by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, respectively. Define

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$$
$$\mu \equiv \frac{m_1m_2}{m_1+m_2}\ ,$$

where μ is the *reduced mass.*
(**a**)
Show that the total kinetic energy $T = T_1 + T_2$ is equal to

$$T = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 + \tfrac{1}{2}M\dot{\mathbf{R}}^2\ ,$$

where $M = m_1 + m_2$.
Solution:
First let's get what we can out of the definition of the CM:

$$\begin{aligned} \mathbf{r} &\equiv \mathbf{r}_1 - \mathbf{r}_2 \\ 0 &\equiv m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \\ \Rightarrow \mathbf{r}_1 &= +\mathbf{r}\frac{m_2}{m_1+m_2} \\ \mathbf{r}_2 &= -\mathbf{r}\frac{m_1}{m_1+m_2}\ . \end{aligned}$$

Using these results to evaluate the kinetic energy,

$$
\begin{aligned}
T &= T_{\text{of CM}} + T_{\text{wrt CM}} \\
T_{\text{of CM}} &= \tfrac{1}{2} M \dot{\mathbf{R}}^2 \\
T_{\text{wrt CM}} &= \tfrac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\mathbf{r}}_2^2 \\
&= \frac{\dot{\mathbf{r}}^2}{2(m_1+m_2)^2}(m_1 m_2^2 + m_2 m_1^2) \\
&= \frac{\dot{\mathbf{r}}^2}{2(m_1+m_2)} m_1 m_2 \\
&\equiv \tfrac{1}{2}\mu \dot{\mathbf{r}}^2 \\
T &= \tfrac{1}{2} M \dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu \dot{\mathbf{r}}^2 \ .
\end{aligned}
$$

(**b**)
About the CM, show that the total angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ is equal to

$$\mathbf{L} = \mu \mathbf{r} \times \dot{\mathbf{r}} \ .$$

Solution:
Here we are concerned only with the angular momentum about the CM.

$$
\begin{aligned}
\mathbf{L} &= \mathbf{L}_1 + \mathbf{L}_2 \\
\mathbf{L}_1 &= m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 \\
\mathbf{L}_2 &= m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2 \ .
\end{aligned}
$$

Using the results from part (**a**) for $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$
\begin{aligned}
\mathbf{L} &= \frac{m_1 m_2^2}{(m_1+m_2)^2}\mathbf{r} \times \dot{\mathbf{r}} + \frac{m_2 m_1^2}{(m_1+m_2)^2}\mathbf{r} \times \dot{\mathbf{r}} \\
&= \frac{m_1 m_2 (m_1+m_2)}{(m_1+m_2)^2}\mathbf{r} \times \dot{\mathbf{r}} \\
&= \frac{m_1 m_2}{(m_1+m_2)}\mathbf{r} \times \dot{\mathbf{r}} \\
&\equiv \mu \mathbf{r} \times \dot{\mathbf{r}} \ .
\end{aligned}
$$

The simplicity of these formulæ explains why the two-particle separation $\mathbf{r}$ and the two-particle reduced mass μ are usually chosen as parameters for analysis of the two-body problem.

3.
Two particles connected by an elastic string of stiffness k and equilibrium length b rotate about their center of mass with angular momentum l. Show that their distances of closest and furthest approach, r_1 and r_2, are related by

$$r_1^2 r_2^2 (r_1 + r_2 - 2b) = (r_1 + r_2) l^2 / k\mu \ ,$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the two-body reduced mass.
Solution:
Remember that a problem with two bodies and a central force can always be treated as a one-body problem: Just replace the mass by the reduced mass μ, and use the separation $\mathbf{r}$ between the bodies as your coordinate. Generically, if r is the distance from the origin and ω is the angular velocity, then the angular momentum l and energy E are

$$
\begin{aligned}
l &= \mu \omega r^2 \\
E &= \tfrac{1}{2}\mu \dot{r}^2 + \tfrac{1}{2}\mu (r\omega)^2 + \tfrac{1}{2} k (r-b)^2 \ ,
\end{aligned}
$$

where the last term is the potential energy in the stretched string. Let's use the equation for l to eliminate ω:

$$E = \frac{1}{2}\left(\mu \dot{r}^2 + \frac{l^2}{\mu r^2} + k(r-b)^2\right) \ .$$

When r is an extremum (at r_1 and r_2), $\dot{r} = 0$, so, setting the energies at r_1 and r_2 equal, and doing some algebra, we get

$$
\begin{aligned}
\frac{l^2}{\mu r_1^2} + k(r_1 - b)^2 &= \frac{l^2}{\mu r_2^2} + k(r_2 - b)^2 \\
\frac{l^2 (r_2^2 - r_1^2)}{\mu r_1^2 r_2^2} &= k\left((r_2 - b)^2 - (r_1 - b)^2\right) \\
r_1^2 r_2^2 (r_1 + r_2 - 2b) &= \frac{l^2}{k\mu}(r_1 + r_2) \ .
\end{aligned}
$$

4.
Determine which of the following forces are conservative, and find the potential energy (within a constant) for those which are:
(**a**)

$$
\begin{aligned}
F_x &= 6abyz^3 - 20bx^3y^2 \\
F_y &= 6abxz^3 - 10bx^4y \\
F_z &= 18abxyz^2 \ .
\end{aligned}
$$

Solution:
Remember that $F_x = -\partial U/\partial x$, and similarly

for F_y and F_z. So (if appropriate constants of integration are chosen) $U = -\int F_x\,dx = -\int F_y\,dy = -\int F_z\,dz$. If those three integrals aren't equal, then $\mathbf{F}$ can't be derived from a potential, and is not conservative. An equivalent statement is that $\mathbf{F}$ is conservative if and only if all the "mixed derivatives" are equal (*i.e.*, if $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$, $\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$, and $\frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$). "Equal mixed derivatives" is the same as "zero curl".

For part (**a**), if you work out all the mixed derivatives, you find that they are all OK (so the force *is* conservative). However, since we have to find U anyway, we'll just do the integrals:

$$\begin{aligned} U &= -\int F_x\,dx \\ &= -(6abxyz^3 - 5bx^4y^2 + A(y,z)) \\ &= -\int F_y\,dy \\ &= -(6abxyz^3 - 5bx^4y^2 + B(x,z)) \\ &= -\int F_z\,dz \\ &= -(6abxyz^3 + C(x,y)) \ , \end{aligned}$$

where A, B, and C are integration constants. (Note that they only have to be constant with respect to the variable of integration.) We can see that if we set $A = B = 0$, and $C = -5bx^4y^2$, we've got it. So

$$U = -6abxyz^3 + 5bx^4y^2 \ .$$

(**b**)

$$\begin{aligned} F_x &= 18abyz^3 - 20bx^3y^2 \\ F_y &= 18abxz^3 - 10bx^4y \\ F_z &= 6abxyz^2 \ . \end{aligned}$$

Solution:
This force is *not* conservative. It can't be, because $\frac{\partial F_x}{\partial z} \neq \frac{\partial F_z}{\partial x}$:

$$\begin{aligned} \frac{\partial F_x}{\partial z} &= 54abyz^2 \ , \text{ but} \\ \frac{\partial F_z}{\partial x} &= 6abyz^2 \ . \end{aligned}$$

(**c**)

$$\mathbf{F} = \hat{\mathbf{x}}F_1(x) + \hat{\mathbf{y}}F_2(y) + \hat{\mathbf{z}}F_3(z) \ .$$

Solution:
This force is conservative, since all mixed derivatives are equal (in fact, they're all zero). The potential is

$$U = -\int F_1(x)\,dx - \int F_2(y)\,dy - \int F_3(z)\,dz \ .$$

5.
A vector field $\mathbf{F}$ is expressed in cylindrical coordinates as follows:

$$\begin{aligned} F_r &= 0 \\ F_\varphi &= k/r \\ F_z &= 0 \ , \end{aligned}$$

where k is a constant.
(**a**)
When $r > 0$, show that $\mathbf{F}$ has zero curl.
Solution:
Transforming $\mathbf{F}$ to Cartesian coordinates,

$$\begin{aligned} F_z &= 0 \\ F_x &= F_r\cos\varphi - F_\varphi\sin\varphi \\ &= 0 - \tfrac{k}{r}\sin\varphi \\ &= \frac{kr\sin\varphi}{r^2} \\ &= -\frac{ky}{x^2+y^2} \ . \end{aligned}$$

Similarly

$$F_y = +\frac{kx}{x^2+y^2} \ .$$

Obviously $(\nabla\times\mathbf{F})_x = (\nabla\times\mathbf{F})_y = 0$ because $\mathbf{F}$ has no z component and no z dependence. The surviving component is

$$\begin{aligned} (\nabla\times\mathbf{F})_z &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \\ &= \frac{k}{x^2+y^2} - \frac{2kx^2}{(x^2+y^2)^2} + \\ &\quad + \frac{k}{x^2+y^2} - \frac{2ky^2}{(x^2+y^2)^2} \\ &= \frac{2k}{x^2+y^2} - \frac{2k(x^2+y^2)}{(x^2+y^2)^2} \\ &= \frac{2k}{x^2+y^2} - \frac{2k}{(x^2+y^2)} \\ &= 0 \ \text{ provided } r \neq 0 \ . \end{aligned}$$

Notice that the final two terms can be considered to cancel only away from $r = 0$, where their denominators vanish and the terms themselves are infinite.

(**b**)
Consider the loop integral $\oint \mathbf{F} \cdot d\mathbf{l}$ counterclockwise around a circular path of fixed radius R. What is the value of this integral?
Solution:
Working in cylindrical coordinates,

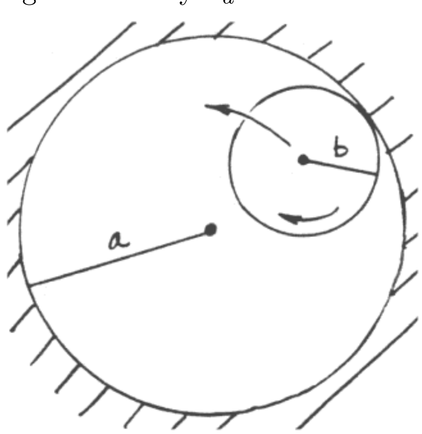
$$\oint \mathbf{F} \cdot d\mathbf{r} = 2\pi R F_\varphi(R) \\ = 2\pi k > 0 \; .$$

(**c**)
Can $\mathbf{F}$ be derived from a single-valued potential U, *e.g.* $\mathbf{F} = -\nabla U$ where $U = U(r, \varphi, z)$? Why or why not?
Solution:
If it were true that $\mathbf{F} = -\nabla U$, then

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint (-\nabla U) \cdot d\mathbf{r} \\ = -U_0 + U_0 \\ = 0 \; ,$$

where U_0 is the single value of U at the starting (and ending) point of the loop integral. This condition is not satisfied from (**b**), so it is *not* true that $\mathbf{F} = -\nabla U$ even though, as shown in part (**a**), $\nabla \times \mathbf{F} = 0$ at finite r.

6.
A hoop of radius b and mass m rolls without slipping within a circular hole of radius $a > b$. About the center of the hole, the point of contact has a uniform angular velocity ω_a.



(**a**)
Find the angular velocity ω_b of the hoop about *its own* center (magnitude and direction).
Solution:
Imagine for a moment that the hoop is moving *while fully* slipping, *i.e.* imagine that the same point on the hoop always makes contact with the hole. In one counterclockwise (CCW) revolution of the point of contact, the hoop would also undergo one CCW revolution.

Now back to the real problem. Take the hoop to be rolling *without* slipping. In one revolution of the point of contact, the point of contact moves a distance $2\pi a$ along the circumference of the hole. Since the circumference of the hoop is smaller, $2\pi b$ instead of $2\pi a$, the fact that it is rolling causes the hoop to rotate *clockwise* by an *extra* number of revolutions equal to $2\pi a/2\pi b$.

Putting the above arguments into an equation, considering the CCW direction to be positive,

$$\omega_b = +\omega_a - \frac{a}{b}\omega_a \; .$$

Here the first term on the RHS is the angular velocity that would result from *full slipping*, while the second term results from the extra revolutions of the hoop due to rolling *without slipping*. Simplifying,

$$\omega_b = -(\tfrac{a}{b} - 1)\omega_a \; .$$

(**b**)
Calculate the kinetic energy T of the hoop.
Solution:
This is a straightforward application of a decomposition rule: the total kinetic energy T is the sum of the kinetic energy $T_{\text{of CM}}$ *of* the CM and the kinetic energy $T_{\text{wrt CM}}$ *with respect to* the CM. Applying it,

$$T = T_{\text{of CM}} + T_{\text{wrt CM}} \\ = \tfrac{1}{2}m(a-b)^2\omega_a^2 + \tfrac{1}{2}mb^2\omega_b^2 \\ = \tfrac{1}{2}m\omega_a^2\big((a-b)^2 + b^2(\tfrac{a}{b} - 1)^2\big) \\ = m\omega_a^2(a-b)^2 \; .$$

In the second line above, we used the fact that the radius of the circle described by the motion of the CM of the hoop is $a-b$.

(**c**)

Obtain the angular momentum L (magnitude and direction) of the hoop relative to the center of the hole.

Solution:

Again this is a straightforward application of a decomposition rule, this time for angular momentum instead of kinetic energy. Applying it, taking the positive direction of $\mathbf{L}$ to be out of the paper,

$$\begin{aligned} L &= L_{\text{of CM}} + L_{\text{wrt CM}} \\ &= (a-b)^2\omega_a + mb^2\omega_b \\ &= m\omega_a\left((a-b)^2 - b^2(\tfrac{a}{b}-1)\right) \\ &= m\omega_a(a^2-2ab+b^2-ab+b^2) \\ &= m\omega_a(a^2-3ab+2b^2) \\ &= m\omega_a(a-b)(a-2b)\ . \end{aligned}$$

(**d**)

Consider your answers to (**b**) and (**c**) in the limit $b\to a$. You should find that both T and L vanish. Is this reasonable? Why or why not?

Solution:

Both T and L contain the factor $(a-b)$, and so they vanish as b approaches a. (Amusingly, L also vanishes when $b=a/2$.) This is perfectly reasonable. Both T and L *must* vanish in the limit $a\to b$; in this limit, the point of contact rotates without the hoop moving at all!

7.

Consider a spherically symmetric distribution $\rho(r)$ of mass density. If the gravitational acceleration, or "gravitational field vector" $\mathbf{g}$, is known to be independent of the radial coordinate r within a spherical volume, find $\rho(r)$ to within a multiplicative constant.

Solution:

Remember: The gravitational acceleration due to the mass of a spherical shell vanishes if you are inside the shell. But if you are outside the shell, then the acceleration is the same as if the mass were all concentrated at the center. In our case, this means that you can find the acceleration g at any distance r from the origin by considering only the mass lying at distance less than r from the origin:

$$\begin{aligned} g(r) &= \frac{GM_{<r}}{r^2} \\ &= \frac{G}{r^2}\int_0^r \rho(r')\,d^3x' \\ &= \frac{4\pi G}{r^2}\int_0^r \rho(r')r'^2\,dr'\ . \end{aligned}$$

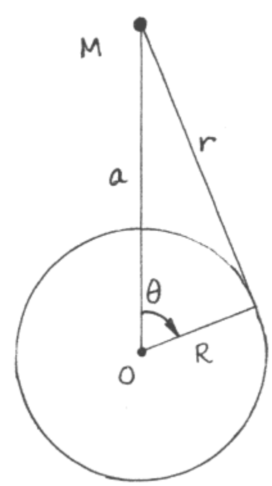
Now, if we want g to be a constant, then we need the integral in this equation to be proportional to r^2 (to cancel the $1/r^2$). If the integral is proportional to r^2, then the integrand is proportional to r. So

$$\begin{aligned} r^2\rho(r) &= kr \qquad \text{where k is a constant} \\ \rho &= k/r\ . \end{aligned}$$

Therefore ρ is proportional to $1/r$. (The constant k is $g/2\pi G$, in case you're interested.)

8.

Consider a point mass that lies outside a spherical surface. Let $\phi(\mathbf{r})$ be the gravitational potential due to the point mass.



Show that the average value of ϕ taken over the spherical surface is the same as the value of ϕ at the center of the sphere. [Since the potential due to an arbitrary mass distribution is the sum of potentials due to point masses, this statement is also true for the gravitational potential due to an arbitrary mass distribution lying outside a spherical surface.]

Solution:
Let a be the distance from the origin to the point mass, and R the radius of the sphere. The average of ϕ over the surface of the sphere is the integral over the whole surface divided by the surface area: $\langle\phi\rangle = \frac{1}{4\pi R^2}\int_{\text{sphere}}(-GM/r)\,dA$. By the law of cosines,

$$r = \sqrt{a^2 + R^2 - 2aR\cos\theta}\ ,$$

and of course the surface area element of the sphere is $dA = R^2 \sin\theta\, d\theta\, d\varphi$, where φ is the "longitudinal" angle around the surface of the sphere.

$$\begin{aligned}
\langle\phi\rangle &= -\frac{GM}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{a^2+R^2-2aR\cos\theta}} \\
u &\equiv \cos\theta \\
\langle\phi\rangle &= -\tfrac{1}{2}GM\int_{-1}^{1}\frac{du}{\sqrt{a^2+R^2-2aRu}} \\
&= \frac{GM}{2aR}\left[\sqrt{a^2+R^2-2aRu}\right]_{-1}^{1} \\
&= -\frac{GM}{2aR}\left(\sqrt{(a+R)^2}-\sqrt{(a-R)^2}\right) \\
&= -\frac{GM}{2aR}\,2R \\
&= -\frac{GM}{a}\ .
\end{aligned}$$

$-GM/a$ is the value of the potential at the center of the sphere, so we've shown what we intended to.